

# Complement for Algebraic differential equations...

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ABSTRACT. We complete the study of some periods of polynomials in  $(n+1)$ -variables with  $(n+2)$ -monomials in computing the behavior of these periods in the natural parameter for such a polynomial.

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# 1 Introduction

This note is a complement to the study in [B.13] of period integrals of non quasi-homogeneous polynomials in  $n + 1$  variables with  $n + 2$  monomials. We focus here on the dependance of these period integrals on the “natural” parameter  $\lambda \in \mathbb{C}^*$  which is the only “free” coefficient of such a polynomial modulo the dilatations of the variables<sup>1</sup>

For that purpose we recall first the fact that for a polynomial function  $f$  depending polynomially of a parameter  $\lambda$  we may define a natural “ $b$ –connection” on the highest  $(f, \lambda)$ –relative de Rham cohomology group of  $f$  which induces the derivation  $\frac{\partial}{\partial \lambda}$  on period integrals. The construction for any holomorphic function depending of a holomorphic parameter is precised in the appendix.

Then we show how to compute explicitly this connection in our specific situation and we obtain a simple partial differential equation for the period integrals associated to any monomial in  $\mathbb{C}[x_0, \dots, x_n]$  when we consider a polynomial of the type

$$f_\lambda(x) = \sum_{j=1}^{n+1} x^{\alpha_j} + \lambda \cdot x^{\alpha_{n+2}} \quad \text{where} \quad \alpha_j \in \mathbb{N}^{n+1}, j \in [1, n+2]$$

with the following assumptions

- i) The  $(n+2, n+2)$ –matrix obtained from  $M := (\alpha_1, \dots, \alpha_{n+2})$  by adding a first line of 1 has rank  $n+2$ .
- ii) The elements  $\alpha_1, \dots, \alpha_{n+1}$  form a basis of  $\mathbb{Q}^{n+1}$ .

Note that the first condition is equivalent to the fact that  $f$  is not quasi-homogeneous, and that the condition ii) is always satisfied assuming i), up to change the order of the monomials (and then to change the parameter  $\lambda$  to  $c \cdot \lambda^m$  for some  $c \in \mathbb{Q}^*$  some  $m \in \mathbb{Z}^*$ ).

## 2 The $\lambda$ –connection.

### 2.1 The general situation.

We consider here a polynomial  $f \in R := \mathbb{C}[x_0, \dots, x_n][\lambda]$  depending polynomially on a parameter  $\lambda$ . We consider on  $R \otimes \Lambda^*(\mathbb{C}^{n+1}) := \Omega_J^*$  the  $\lambda$ –relative de Rham complex, where  $(\mathbb{C}^{n+1})^* := \bigoplus_{i=0}^n \mathbb{C} \cdot dx_i$ , and we denote  $d_J$  its differential.

We shall denote by  $\mathcal{A}$  the unitary (non commutative) algebra generated by  $a$  and  $b$  with the commutation relation  $a \cdot b - b \cdot a = b^2$  and by  $\mathcal{A}[\lambda] := \mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}[\lambda]$  with its natural structure of algebra for which  $\lambda$  commutes with  $a$  and  $b$ .

Then the quotient  $E_f := \Omega_J^{n+1} / d_J f \wedge d_J \Omega_J^{n-1}$  has a natural left  $\mathcal{A}[\lambda]$ –module structure defined by

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<sup>1</sup>In our hypothesis we may assume that all monomials have coefficient 1 excepted the last one up to a linear diagonal change of variable.

- The action of  $a$  is given by the multiplication by  $f$ .
- The action of  $b$  is given by  $d/f \wedge d_f^{-1}$ .

Remark that for fixed  $\lambda$ , assuming that  $f_\lambda$  has an isolated singularity at 0, the  $b$ -completion  $E_f \otimes_{\mathbb{C}[b]} \mathbb{C}[[b]]$  is the usual (formal) Brieskorn module associated to  $f_\lambda$  at 0. For a given monomial  $\mu \in \mathbb{C}[x_0, \dots, x_n]$  the decomposition theorem of [B.13] (theorem 3.1.2) applies to the quotient  $\tilde{\mathcal{A}}/\tilde{\mathcal{A}}.P(\mu)$  where  $\tilde{\mathcal{A}}$  is the  $b$ -completion of  $\mathcal{A}$  and where  $P(\mu)$  is the element in  $\mathcal{A}$  constructed in the theorem 1.2.1 of [B.13]. Then  $P_d(\mu)$  is a (left-)multiple of the Bernstein element of  $\tilde{\mathcal{A}}.[\mu.dx]$  in  $E_f \otimes_{\mathbb{C}[b]} \mathbb{C}[[b]]$  and it determines a finite set of possible eigenvalues for the monodromies around  $s = 0$  for the period integrals ( $\lambda$  fixed)

$$\varphi_\lambda(s) = \int_{\gamma_{\lambda,s}} \frac{\mu.dx}{d/f}$$

for any horizontal family  $\gamma_{\lambda,s}$  of compact  $n$ -dimensional cycles in the fibers of  $f_\lambda$ .

It is important to remark that if  $f_\lambda$  has a non isolated singularity at the origin, despite the fact that there is no finiteness for the  $\mathbb{C}[[b]]$ -module  $E_f \otimes_{\mathbb{C}[b]} \mathbb{C}[[b]]$ , the conclusion above is still valid because the quotient  $\tilde{\mathcal{A}}/\tilde{\mathcal{A}}.P(\mu)$ , and so its image in  $E_f \otimes_{\mathbb{C}[b]} \mathbb{C}[[b]]$ , is a finite type  $\mathbb{C}[[b]]$ -module. Then the product decomposition  $P_d = (a - r_1.b) \dots (a - r_d.b)$ , where  $r_1, \dots, r_d$  are (explicitly computable) rational numbers, gives that the set  $\{e^{2i\pi.r_1}, \dots, e^{2i\pi.r_d}\}$  contains the spectrum of these monodromies (counting multiplicities).

QUESTION. Is it true that  $P_d$  is equal to the Bernstein element<sup>2</sup> of the Brieskorn module  $\tilde{\mathcal{A}}.[\mu.dx]$  in  $E_f \otimes_{\mathbb{C}[b]} \mathbb{C}[[b]]$  when  $f_\lambda$  has an isolated singularity at the origin ?  $\square$

**Proposition 2.1.1** *There exists a  $\mathbb{C}$ -linear operator  $\nabla : E_f \rightarrow E_f$  with the following properties :*

1. For  $\omega = d/\xi \in \Omega_f^{n+1}$  we have  $\nabla([\omega]) = [d/f \wedge \frac{\partial \xi}{\partial \lambda} - \frac{\partial f}{\partial \lambda}.\omega]$ .
2. The map  $b^{-1}.\nabla$  well defined on  $b.\tilde{E}_f$  where  $\tilde{E}_f := E_f/(b - \text{torsion})$ , with value in  $\tilde{E}_f$ , commutes with  $a$  and  $b$  and is a  $\lambda$ -connection.

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<sup>2</sup>For a  $(a,b)$ -module  $E$  with one generator as a  $\tilde{\mathcal{A}}$ -module, the relation between its Bernstein element  $P_d \in \mathcal{A}$  and its Bernstein polynomial  $B$  is given by the formula (see [B.09])

$$(-b)^d.B(-b^{-1}.a) = P_d \quad \text{where } d \text{ is the rank of } E.$$

3. If  $(\gamma_{s,\lambda})_{(s,\lambda) \in S \times \Omega}$  is a horizontal family of compact  $n$ -cycles in the fibers of  $(f, \lambda)$  over an open set in  $\mathbb{C} \times \mathbb{C}^* \setminus C(f, \lambda)$  where  $C(f, \lambda)$  is the set of critical values of the map  $(f, \text{id}) : \mathbb{C}^{n+1} \times \mathbb{C}^* \rightarrow \mathbb{C} \times \mathbb{C}^*$ , we have for any  $\omega \in \Omega_{/}^{n+1}$  the equality

$$\frac{\partial}{\partial s} \frac{\partial}{\partial \lambda} \left[ \int_{\gamma_{s,\lambda}} \frac{\omega}{d_{/}f} \right] = \int_{\gamma_{s,\lambda}} \frac{\nabla(\omega)}{d_{/}f}.$$

PROOF. Remark first that  $\nabla$  is well defined because for  $\xi = d_{/}\eta$  we have

$$\nabla(d_{/}\xi) = d_{/}f \wedge \frac{\partial(d_{/}\eta)}{\partial \lambda} = d_{/}f \wedge d_{/}\left(\frac{\partial \eta}{\partial \lambda}\right)$$

so it induces 0 in  $E_f$ .

Let  $\omega = d_{/}\xi \in \Omega_{/}^{n+1}$  and let  $d_{/}\eta = d_{/}f \wedge \xi$ . Then we have

$$\begin{aligned} \nabla(b.[\omega]) &= \nabla(d_{/}\eta) = d_{/}f \wedge \frac{\partial \eta}{\partial \lambda} - \frac{\partial f}{\partial \lambda} \cdot d_{/}\eta \\ &= d_{/}f \wedge \left( \frac{\partial \eta}{\partial \lambda} - \frac{\partial f}{\partial \lambda} \cdot \xi \right) = b \left[ d_{/} \left( \frac{\partial \eta}{\partial \lambda} - \frac{\partial f}{\partial \lambda} \cdot \xi \right) \right] \\ &= b \left[ \frac{\partial(d_{/}f)}{\partial \lambda} \wedge \xi + d_{/}f \wedge \frac{\partial \xi}{\partial \lambda} - \frac{\partial f}{\partial \lambda} \cdot d_{/}\xi - d_{/} \left( \frac{\partial f}{\partial \lambda} \right) \wedge \xi \right] \\ &= b \left[ d_{/} \left( \frac{\partial f}{\partial \lambda} \right) \cdot \xi + \nabla(d_{/}\xi) - d_{/} \left( \frac{\partial f}{\partial \lambda} \right) \wedge \xi \right] = b[\nabla(d_{/}\xi)] \end{aligned}$$

as  $d_{/}$  and  $\frac{\partial}{\partial \lambda}$  commute. So we have  $b \cdot \nabla = \nabla \cdot b$ .

We have also

$$\begin{aligned} \nabla(a.[\omega]) &= \nabla(f \cdot d_{/}\xi) = \nabla(d_{/}(f \cdot \xi)) - \nabla(d_{/}f \wedge \xi) \\ &= d_{/}f \wedge \frac{\partial(f \cdot \xi)}{\partial \lambda} - \frac{\partial f}{\partial \lambda} \cdot f \cdot d_{/}\xi - \frac{\partial f}{\partial \lambda} \cdot d_{/}f \wedge \xi - \nabla(b.[\omega]) \\ &= a \cdot \nabla([\omega]) - b \cdot \nabla([\omega]). \end{aligned}$$

This implies the equality  $a \cdot b^{-1} \cdot \nabla = b^{-1} \cdot \nabla \cdot a$  as  $\mathbb{C}$ -linear maps from  $b \cdot \tilde{E}_f$  to  $\tilde{E}_f$ .

Note that the equalities  $\nabla \cdot b = b \cdot \nabla$  and  $\nabla \cdot a = (a - b) \cdot \nabla$  as  $\mathbb{C}$ -endomorphisms of  $E_f$  are more precise than the relations above.

Finally let  $\varphi \in \mathbb{C}[\lambda]$  then we have

$$\begin{aligned} \nabla(\varphi \cdot d_{/}\xi) &= \nabla(d_{/}(\varphi \cdot \xi)) = d_{/}f \wedge \frac{\partial \varphi \cdot \xi}{\partial \lambda} - \frac{\partial f}{\partial \lambda} \cdot \varphi \cdot d_{/}\xi \\ &= \frac{\partial \varphi}{\partial \lambda} \cdot (d_{/}f \wedge \xi) + \varphi \cdot \nabla(d_{/}\xi) = \frac{\partial \varphi}{\partial \lambda} \cdot b[d_{/}\xi] + \varphi \cdot \nabla(d_{/}\xi) \end{aligned}$$

and this shows that  $b^{-1}.\nabla$  is a  $\lambda$ -connection.

Note again that we proved the equality in  $E_f$  :  $\nabla(\varphi.\omega) = \frac{\partial \varphi}{\partial \lambda}.b.\omega + \varphi.\nabla(\omega)$  valid for  $\varphi \in \mathbb{C}[\lambda]$  and  $\omega \in E_f$  which is more precise.

To prove the point 3. of the statement consider  $\xi \in \Omega_f^n$  and let  $d$  be the total de Rham differential (in  $x$  and  $\lambda$ ). We have

$$d\xi = d\lambda \wedge \frac{\partial \xi}{\partial \lambda} + d_{/f}\xi \quad \text{and} \quad df = d\lambda \cdot \frac{\partial f}{\partial \lambda} + d_{/f}f.$$

Assume we can write  $d\xi = d\lambda \wedge v + d_{/f}f \wedge u$  with  $u, v \in \Omega_f^n$ . Then we obtain

$$d\xi = d\lambda \wedge (v - \frac{\partial f}{\partial \lambda}.u) + d_{/f}f \wedge u \quad \text{with} \quad u = \frac{d_{/f}\xi}{d_{/f}f} \quad \text{and} \quad v = \frac{\partial \xi}{\partial \lambda}.$$

If  $(\gamma_{s,\lambda})$  is a horizontal family of compact  $n$ -cycles in the fibers of the map  $(f, \text{id}) : \mathbb{C}^{n+1} \times \mathbb{C}^* \rightarrow \mathbb{C} \times \mathbb{C}^*$ , we shall have

$$d\left(\int_{\gamma_{s,\lambda}} \xi\right) = \left[\int_{\gamma_{s,\lambda}} (v - \frac{\partial f}{\partial \lambda}.u)\right].d\lambda + \left[\int_{\gamma_{s,\lambda}} u\right].ds.$$

So, has the chain  $\cup_{s,\lambda} \gamma_{s,\lambda}$  is proper and without  $\lambda$ -relative boundary we obtain

$$\frac{\partial}{\partial s} \int_{\gamma_{s,\lambda}} \xi = \int_{\gamma_{s,\lambda}} \frac{d_{/f}\xi}{d_{/f}f} \quad \text{and} \quad \frac{\partial}{\partial \lambda} \int_{\gamma_{s,\lambda}} \xi = \int_{\gamma_{s,\lambda}} \left(\frac{\partial \xi}{\partial \lambda} - \frac{\partial f}{\partial \lambda} \cdot \frac{d_{/f}\xi}{d_{/f}f}\right)$$

Now consider  $\omega \in \Omega_f^{n+1}$  and write  $\omega = d_{/f}\xi$ . Then  $b[\omega] = [d_{/f}f \wedge \xi]$  and we have

$$\int_{\gamma_{s,\lambda}} \frac{b[\omega]}{d_{/f}f} = \int_{\gamma_{s,\lambda}} \xi \quad \text{and} \quad \frac{\partial}{\partial s} \int_{\gamma_{s,\lambda}} \frac{b[\omega]}{d_{/f}f} = \int_{\gamma_{s,\lambda}} \frac{\omega}{d_{/f}f}.$$

So we conclude that we have

$$\frac{\partial}{\partial \lambda} \int_{\gamma_{s,\lambda}} \frac{\omega}{d_{/f}f} = \frac{\partial}{\partial \lambda} \int_{\gamma_{s,\lambda}} \xi = \int_{\gamma_{s,\lambda}} \left(\frac{\partial \xi}{\partial \lambda} - \frac{\partial f}{\partial \lambda} \cdot \frac{d_{/f}\xi}{d_{/f}f}\right) = \frac{\partial}{\partial \lambda} \int_{\gamma_{s,\lambda}} \frac{\nabla[\omega]}{d_{/f}f} \quad \blacksquare$$

## 2.2 The case of a polynomial with $n+2$ monomials in $n+1$ variables.

So we consider now the case were  $f := \sum_{j=1}^{n+2} m_j$  where  $m_j := x^{\alpha_j}$   $j \in [1, n+1]$  and  $m_{n+2} := \lambda.x^{\alpha_{n+2}}$  with the following hypotheses (see [B. 13]) : the rank of the square matrix  $M' := (\alpha_1, \dots, \alpha_{n+1})$  is  $n+1$  and the rank of the square matrix  $\tilde{M}$  obtained by adding a first line of 1 to the matrix  $M := (\alpha_1, \dots, \alpha_{n+2})$  is  $n+2$ .

Recall that if we write (with a minimal positive integer  $r$ )  $r \cdot \alpha_{n+2} = \sum_{j=1}^{n+1} p_j \cdot \alpha_j$  where  $p_1, \dots, p_{n+1}$  are in  $\mathbb{Z}$ , and if we define

$$d = \inf \left\{ r - \sum_{j, p_j \leq 0} p_j, \sum_{j, p_j > 0} p_j \right\} \quad \text{and} \quad d + h = \sup \left\{ r - \sum_{j, p_j \leq 0} p_j, \sum_{j, p_j > 0} p_j \right\}$$

there exists an element  $P$  in  $A[\lambda, \lambda^{-1}]$  of the form

$$P := P_{d+h} + c \cdot \lambda^{\pm r} \cdot P_d$$

which annihilated the class  $[dx]$  in  $E_f$ , where  $P_{d+h}$  and  $P_d$  are homogeneous elements in  $\mathcal{A}$ , respectively of degree  $d+h$  and  $d$  which are monic in  $a$  with rational coefficients<sup>3</sup>, and where  $c$  is in  $\mathbb{Q}^*$ . The sign in the exponent of  $\lambda$  will be precised in the proof of the proposition 2.2.2.

Recall also that in this situation the  $\mathcal{A}[\lambda]$  module generated by the class  $[dx]$  in  $E_f$  is exactly the image in  $E_f$  of  $\mathbb{C}[m_1, \dots, m_{n+2}][\lambda].dx \subset \Omega_{/}^{n+1}$  with  $m_j = x^{\alpha_j}$  with  $j \in [1, n+1]$  and  $m_{n+2} = \lambda \cdot x^{\alpha_{n+2}}$ .

Our next result uses the following easy lemma:

**Lemme 2.2.1** *Let  $Q \in \mathcal{A}$  a homogeneous element in  $(a, b)$  of degree  $k$ . Then for any  $\lambda \in \mathbb{C}$  we have :*

$$b.Q.b^{-1} \cdot (a - \lambda.b) = (a - (\lambda + k).b).Q.$$

PROOF. Remark first that the map  $\mathcal{A} \rightarrow \mathcal{A}$  sending  $x \in \mathcal{A}$  to  $b.x.b^{-1}$  is well defined and bijective thanks to the following facts :  $b$  is injective and  $b.\mathcal{A} = \mathcal{A}.b$ . We shall prove the lemma by induction  $k$ . As the case  $k = 0$  is obvious, assume that the lemma is proved for  $k < k_0$  where  $k_0 \geq 1$  and consider an homogeneous element  $Q$  of degree  $k_0$ . We may assume<sup>4</sup> that  $Q = b.R$  or that we may find  $\mu \in \mathbb{C}$  such that  $Q = (a - \mu.b).R$ , where  $R$  is homogeneous of degree  $k_0 - 1$ . In the first case we have, using the induction hypothesis :

$$b.b.R.b^{-1} \cdot (a - \lambda.b) = b.(a - (\lambda + k_0 - 1).b).R = (a - (\lambda + k_0).b).b.R = (a - (\lambda + k_0).b).Q.$$

In the second case we have, using the induction hypothesis :

$$\begin{aligned} b.(a - \mu.b).R.b^{-1} \cdot (a - \lambda.b) &= (a - (\mu + 1).b).b.R.b^{-1} \cdot (a - \lambda.b) \\ &= (a - (\mu + 1).b).(a - (\lambda + k_0 - 1).b).R \\ &= (a - (\lambda + k_0).b).(a - \mu.b).R \\ &= (a - (\lambda + k_0).b).Q. \end{aligned}$$

■

<sup>3</sup>More is proved in [B.13] :  $P_{d+h}$  and  $P_d$  factorize in product of  $(a - r_j.b)$  with  $r_j \in \mathbb{Q}$ .

<sup>4</sup>Recall that any homogeneous element in  $\mathcal{A}$  which is monic in  $a$  factorizes as a product of linear factors  $(a - r_i.b)$ , where the  $r_i$  are complex numbers; see [B.09].

**Proposition 2.2.2** *Let  $\mu$  be a monomial of degree  $k$  in  $\mathbb{C}[x_0, \dots, x_{n+1}]$ . Then we have in  $E_f$  the relation*

$$\nabla([\mu]) = \frac{-1}{\lambda} \cdot (\sigma \cdot a + (\tau - k \cdot \sigma) \cdot b)[\mu]$$

where  $\sigma, \tau$  are defined by the relation  $m_{n+2} \cdot [\mu] = (\sigma \cdot a + \tau \cdot b)[\mu]$ . Moreover the value<sup>5</sup> of  $\sigma$  is  $\pm r/h$  so it does not depend on the choice of the monomial  $\mu$ .

As a consequence, if we have on an open set  $S \times \Omega$  in  $\mathbb{C}^* \times \mathbb{C}^*$ , a horizontal family  $(\gamma_{s,\lambda})_{(s,\lambda) \in S \times \Omega}$  of compact  $n$ -cycles in the fibers of the map  $\mathbb{C}^{n+1} \times \mathbb{C}^* \rightarrow \mathbb{C} \times \mathbb{C}^*$  defined by  $(x, \lambda) \mapsto (f_\lambda(x), \lambda)$ , the holomorphic function

$$(s, \lambda) \mapsto \varphi(s, \lambda) := \int_{\gamma_{s,\lambda}} \frac{\mu \cdot dx}{d/f}$$

satisfies the partial differential equation

$$-\lambda \cdot \frac{\partial}{\partial \lambda} \frac{\partial}{\partial s} \varphi = \sigma \cdot \frac{\partial(s \cdot \varphi)}{\partial s} + (\tau - k) \cdot \varphi$$

on  $S \times \Omega$ . ■

PROOF. As we have  $\lambda \cdot \nabla([1]) = -m_{n+2}$  in  $E_f$  and as we know that there exist  $\sigma, \tau$  in  $\mathbb{Q}$  such that  $(\sigma \cdot a + \tau \cdot b)[1] = m_{n+2}$  for the case  $\mu = 1$  the only thing to prove is the computation of  $\sigma$ .

Using the Cramer system with matrix  $(n+2, n+2)$  obtained by adding a first line of 1 to the matrix  $M := (\alpha_1, \dots, \alpha_{n+2})$ , computing  $a[1]$  and the  $b_i[1]$  we find that  $\sigma$  is the coefficient  $(n+2, 1)$  in the matrix  $\tilde{M}^{-1}$ . Let  $M'$  be the principal  $(n+1, n+1)$  minor of  $\tilde{M}$ . This implies that

$$\sigma = (-1)^{n+1} \frac{\det(M')}{\det(\tilde{M})}.$$

But using the relation  $\alpha_{n+2} = \sum_{j=1}^{n+1} \frac{p_j}{r} \cdot \alpha_j$  we obtain

$$\det(\tilde{M}) = (-1)^{n+1} \cdot \left(1 - \sum_{j=1}^{n+1} \frac{p_j}{r}\right) \cdot \det(M')$$

so we conclude that

$$\sigma = \frac{r}{r - \sum_{j=1}^{n+1} p_j}.$$

Now we have two cases :

i)  $r - \sum_{p_j < 0} p_j = d + h > \sum_{p_j > 0} p_j = d$ . Then  $r - \sum_{j=1}^{n+1} p_j = (d + h) - d = h$ .

So  $\sigma = r/h$ , and the exponent of  $\lambda$  in  $P$  is  $r$ .

---

<sup>5</sup>The sign is precised in the proof and only depends on  $\alpha_1, \dots, \alpha_{n+2}$ .

- ii)  $\sum_{p_j > 0} p_j = d + h > r - \sum_{p_j < 0} p_j = d$ . Then  $r - \sum_{j=1}^{n+1} p_j = d - (d + h) = -h$ .  
 So  $\sigma = -r/h$ , and the exponent of  $\lambda$  in  $P$  is  $-r$ .

Consider now the case of a degree  $k$  monomial  $\mu \in \mathbb{C}[x_0, \dots, x_n]$ . Then there exists again  $\sigma', \tau'$  in  $\mathbb{Q}$  such that  $(\sigma'.a + \tau'.b)[\mu] = [m_{n+2}.\mu]$  in  $E_f$ . As  $a[\mu], (\beta_i + 1).b[\mu], i \in [0, n]$ , where  $\beta_i$  is the degree in  $x_i$  of  $\mu := x^\beta$ , are again given from the  $[m_j.\mu], j \in [1, n+2]$  by the same Cramer system, we conclude that  $\sigma' = \sigma$ . To conclude the proof it is enough to apply the proposition 2.1.1.  $\blacksquare$

Note that in the case i) above  $P := P_{d+h} + c.\lambda^r.P_d$  annihilated  $[\mu]$  in  $E_f$  and in the case ii) we have  $P := P_{d+h} + c.\lambda^{-r}.P_d$ .

The lemma 2.2.1 gives that  $\lambda.\nabla(P.[\mu]) = -(\sigma.a + (\tau' - k.\sigma).b).P[\mu]$  which makes explicit the fact that  $\lambda.\nabla$  is well defined on  $\mathcal{A}[\lambda].[\mu] \subset E_f$ .

REMARK. Recall that in [B.13] we have built in an explicit way a differential equation in  $s \in S$ , depending in a very simple and concrete way on  $\lambda \in \mathbb{C}^*$  which is satisfied by  $\varphi$ . So it is easy to see that the knowledge of a formal asymptotic expansion when  $s$  goes to 0 in  $S^6$  for a given  $\lambda_0$ , of the type

$$\varphi(\lambda_0, s) \simeq \sum_{i,j} C_{i,j}.s^{\rho_i}.(\text{Log}s)^j$$

where  $\rho_1, \dots, \rho_I$  are in  $-1 + \mathbb{Q}^{*+}$ ,  $j \in [0, n]$  are integers and  $C_{i,j}$  are in  $\mathbb{C}[[s]]$ , determines (uniquely) via the partial differential equation above, a formal expansion of the same type for each given  $\lambda \in \Omega$ , whose coefficients  $C_{i,j}^\lambda$  are polynomials in  $\text{Log}\lambda$  easily computable from the coefficients  $C_{i,j}^{\lambda_0} := C_{i,j}$  of the asymptotic expansion at the initial value  $\lambda_0$  of  $\lambda$ . This is described in the following lemma.

**Lemme 2.2.3** *Let  $\Omega$  be a simply connected domain in  $\mathbb{C}^*$ . Let  $(\rho_i)_{i \in I}$  be a finite collection of rational numbers strictly bigger than  $-1$ . Assume that the formal power serie*

$$\varphi_\lambda := \sum_{k=0}^N \sum_{i \in I} \sum_{m \geq 0} c_m^{i,k}(\lambda).s^{m+\rho_i}.(\text{Log}s)^k / k!$$

where  $c_m^{i,k}$  are holomorphic functions in  $\Omega$ , satisfies the partial differential equation

$$\lambda. \frac{\partial}{\partial \lambda} \frac{\partial}{\partial s} \varphi_\lambda = \alpha.s \frac{\partial(\varphi_\lambda)}{\partial s} + \beta.\varphi_\lambda$$

for each  $\lambda \in \Omega$ . Then for each  $i, k$  fixed, the function  $c_m^{i,k}$  is a polynomial in  $\text{Log}\lambda$  of degree  $\leq m$  for each  $m$ . Moreover the collection of numbers  $c_m^{i,k}(\lambda_0)$  for a given  $\lambda_0 \in \Omega$  determines uniquely these polynomials.

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<sup>6</sup>This is always the case when  $S$  contains an open sector with edge at the origin.



PROOF. The partial differential equation implies the following recursion relation for each  $i, k, m$  :

$$(m + \rho_i + 1) \cdot \lambda \cdot \frac{\partial c_{m+1}^{i,k}(\lambda)}{\partial \lambda} + \lambda \cdot \frac{\partial c_{m+1}^{i,k+1}(\lambda)}{\partial \lambda} = (\alpha \cdot (m + \rho_i) + \beta) \cdot c_m^{i,k}(\lambda) + \alpha \cdot c_m^{i,k+1}(\lambda)$$

We shall make a descending induction on  $k$ . For  $k = N$  the recursion relation reduces to

$$(m + \rho_i + 1) \cdot \lambda \cdot \frac{\partial c_{m+1}^{i,N}(\lambda)}{\partial \lambda} = (\alpha \cdot (m + \rho_i) + \beta) \cdot c_m^{i,N}(\lambda)$$

and an easy induction on  $m \geq 0$  gives our assertion for  $k = N$ .

Assuming the statement proved for  $k + 1$  a simple quadrature in  $\lambda$  implies the case  $k$ . ■

### 3 Two families of examples with $d = 2$ and $h = 1$ .

#### 3.1 The family $x^{2u} + y^{2v} + z^{2w} + \lambda \cdot x^u \cdot y^v \cdot z^w$ .

The condition to be in our situation is  $u \cdot v \cdot w > 0$ . Then we have the relation  $m_4^2 = \lambda^2 \cdot m_1 \cdot m_2 \cdot m_3$  and it shows that  $d = 2$  and  $h = 1$ .

Note that the only singularity of  $f$  in  $\{f = 0\}$  is the origine.

To compute  $P := P_3 + c \cdot \lambda^{-2} \cdot P_2$  which annihilates  $[1]$  is not difficult. We find

$$\begin{aligned} P = & (a - (2 + \frac{u+v}{2u \cdot v}) \cdot b) (a - (1 + \frac{u+w}{2u \cdot w}) \cdot b) (a - (\frac{v+w}{2v \cdot w}) \cdot b) + \\ & - 4\lambda^{-2} \cdot (a - (\frac{3}{2} + \frac{u \cdot v + v \cdot w + w \cdot u}{2u \cdot v \cdot w}) \cdot b) (a - \frac{u \cdot v + v \cdot w + w \cdot u}{2u \cdot v \cdot w} \cdot b). \end{aligned}$$

In this case we have

$$\lambda \cdot \nabla([1]) = 2 \cdot (a - (\frac{u \cdot v + v \cdot w + w \cdot u}{2u \cdot v \cdot w}) \cdot b) [1] = -m_4.$$

Here we are in the case ii) above (so  $\sigma = -2$ ).

Let me illustrate this family on a simple example :  $f = x^4 + y^4 + z^2 + \lambda \cdot x^2 \cdot y^2 \cdot z$  corresponding to  $u = v = 2, w = 1$ . In this case we find

$$P := (a - \frac{5}{2} \cdot b) [(a - \frac{7}{4} \cdot b) (a - \frac{3}{4} \cdot b) - 4 \cdot \lambda^{-2} \cdot (a - b)] \quad \text{and} \quad \lambda \cdot \nabla([1]) = 2(a - b)[1].$$

#### 3.2 The family $x^{2p} \cdot z^u + y^{2q} \cdot z^v + z^{u+v} + \lambda \cdot x^p \cdot y^q$ .

The condition to be in our situation is  $p \cdot q \cdot (u + v) > 0$ . Note that the singularity at the origine is not isolated in general in these cases. We have here the equality

$$2 \cdot \alpha_4 = \alpha_1 + \alpha_2 - \alpha_3$$

The relation which determines  $P$  annihilating  $[1]$  is given by  $m_4^2.m_3 = \lambda^2.m_1.m_2$ , so  $r = 2, d = 2, h = 1$  and we are in the case i).

The computation of  $P$  gives

$$P = (a - (2 + \frac{p+q}{2p.q}).b)(a - (\frac{1}{2} + \frac{p.u+q.v+2p.q}{2p.q.(u+v)}.b)(a - (\frac{p.u+q.v+2p.q}{2p.q.(u+v)}.b) + \\ - 4\lambda^2.(a - (1 + \frac{p.u+q.v+2p.q+p.(u+v)}{2p.q.(u+v)}.b)(a - (\frac{p.u+q.v+2p.q+q.(u+v)}{2p.q.(u+v)}.b)).$$

The computation of  $(\sigma, \tau)$  such that  $(\sigma.a + \tau.b)[1] = m_4$  is easy and it gives

$$\lambda.\nabla([1]) = -(2.a - \frac{p.u+q.v+2p.q}{p.q.(u+v)}.b) = -m_4.$$

Again let me illustrated by an example : for  $p = q = 2$  and  $u = v = 1$  so for  $f = x^4.z + y^4.z + z^2 + \lambda.x^2.y^2$ . We find

$$P := (a - \frac{5}{2}.b)(a - \frac{5}{4}.b)(a - \frac{3}{4}.b) - 4\lambda^2.(a - 2b)(a - b) \quad \text{and} \quad \lambda.\nabla([1]) = -2(a - \frac{3}{4}.b)[1]$$

## 4 Appendix

It is interesting to remark that the proposition 2.1.1 is a special case in a specific algebraic setting of a general result on the the filtered Gauss-Manin connexion of a holomorphic function depending holomorphically of a parameter. This is the goal of this appendix to precise this point.

Let  $M$  be a complex manifold,  $D$  an open disc in  $\mathbb{C}$  and let  $f : D \times M \rightarrow \mathbb{C}$  be a holomorphic function. Denote  $K_{\lambda}^p := \text{Ker}[d_{\lambda}f \wedge; \Omega_{\lambda}^p \rightarrow \Omega_{\lambda}^{p+1}]$  for  $p \geq 2$  and  $K_{\lambda}^1 := \text{Ker}[d_{\lambda}f \wedge : \Omega_{\lambda}^1 \rightarrow \Omega_{\lambda}^2] / \mathcal{O}.d_{\lambda}f$  where  $d_{\lambda}f$  is the  $\lambda$ -relative differential of  $f$  and  $\Omega_{\lambda}^p$  the sheaf of  $\lambda$ -relative holomorphic  $p$ -forms (compare with [B.08]).

Denote by  $(K_{\lambda}^{\bullet}, d_{\lambda})$  the topological restriction of the  $\lambda$ -relative de Rham complex (defined above) for the map  $(\lambda, x) \mapsto (\lambda, f(\lambda, x))$ , to the analytic subset

$$Z := \{d_{\lambda}f = 0\}$$

and let  $\mathcal{H}^p$  the  $p$ -th cohomology sheaf of this complex. Recall that these cohomogy sheaves have a natural structure of left  $\mathcal{A}[\lambda]$ -modules with the action of  $a$  given by the multiplication by  $f$  and with the action of  $b$  defined by  $d_{\lambda}f \wedge d_{\lambda}^{-1}$ .

**Proposition 4.0.1** *There exists a natural graded map  $\nabla^{\bullet} : \mathcal{H}^{\bullet} \rightarrow \mathcal{H}^{\bullet}$  with the following properties :*

1. For  $\omega = d_{\lambda}\xi \in K_{\lambda}^{p+1} \cap \text{Ker } d_{\lambda}$  we have  $\nabla([\omega]) = [d_{\lambda}f \wedge \frac{\partial \xi}{\partial \lambda} - \frac{\partial f}{\partial \lambda}.\omega]$ .

2. The map  $b^{-1}.\nabla$  well defined on  $b.\tilde{\mathcal{H}}^\bullet$  where  $\tilde{\mathcal{H}}^\bullet := \mathcal{H}^\bullet/(b\text{-torsion})$ , with value in  $\tilde{\mathcal{H}}^\bullet$ , commutes with  $a$  and  $b$  and is a  $\lambda$ -connection.
3. If  $(\gamma_{s,\lambda})_{(s,\lambda) \in S \times \Omega}$  is a horizontal family of compact  $p$ -cycles in the fibers of  $(f, \lambda)$  over an open set in  $D \times M \setminus C(f, \lambda)$  where  $C(f, \lambda)$  is the set of critical values of the map  $(\text{id}, f) : D \times M \rightarrow D \times \mathbb{C}$ , we have for any  $\omega \in K^{p+1}_/ \cap \text{Ker } d_/_$  the equality

$$\frac{\partial}{\partial s} \frac{\partial}{\partial \lambda} \left[ \int_{\gamma_{s,\lambda}} \frac{\omega}{d_/_f} \right] = \int_{\gamma_{s,\lambda}} \frac{\nabla(\omega)}{d_/_f}.$$

PROOF. First remark that if  $\omega = d_/_\xi \in K^{p+1}_/$  with  $d_/_f \wedge \xi = 0$ , we have  $d_/_(\frac{\partial f}{\partial \lambda}) \wedge \xi + d_/_f \wedge \frac{\partial \xi}{\partial \lambda} = 0$  so  $\nabla(d_/_\xi) = -d_/_(\frac{\partial f}{\partial \lambda}.\xi)$  is in  $d_/_K^p$ . Now for  $\omega = d_/_\xi \in K^{p+1}_/$  we have  $d_/_f \wedge \nabla(d_/_\xi) = 0$  and

$$d_/_(\nabla(d_/_\xi)) = -d_/_f \wedge d_/_(\frac{\partial \xi}{\partial \lambda}) - d_/_(\frac{\partial f}{\partial \lambda}) \wedge d_/_\xi$$

and we obtain that

$$d_/_(\nabla(d_/_\xi)) = -\frac{\partial}{\partial \lambda}(d_/_f \wedge \omega) = 0$$

using the fact that  $\frac{\partial}{\partial \lambda}(d_/_f \wedge \omega) \equiv 0$ .

The proof of the other statements are analogous to the corresponding ones in proposition 2.1.1. ■

REMARKS.

1. As above we have a more precise formulation for the properties in assertion 2. of the proposition above with the following relations in  $\mathcal{H}^\bullet$

$$\begin{aligned} \nabla(\varphi.\omega) &= \frac{\partial \varphi}{\partial \lambda}.b.\omega + \varphi.\nabla(\omega) \quad \text{for } \varphi \in \mathcal{O}_\lambda \quad \text{and } \omega \in \mathcal{H}^\bullet \\ b.\nabla &= \nabla.b \quad \text{and} \quad \nabla.a = (a - b).\nabla. \end{aligned}$$

2. The generalization of this proposition to several holomorphic parameters is immediate. □

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